

Two-soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions

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Abstract

An explicit two-soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions is derived, demonstrating details of interactions between two bright solitons, two dark solitons, as well as one bright soliton and one dark soliton. Shifts of soliton positions due to collisions are analytically obtained, which are irrespective of the bright or dark characters of the participating solitons.

Key words: DNLS equation, solitons, nonvanishing boundary conditions

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The derivative nonlinear Schrödinger (DNLS) equation is an integrable model describing various nonlinear waves such as nonlinear Alfvén waves in space plasma(see, e.g., [1,2,3,4,5,6,7]), sub-picosecond pulses in single mode optical fibers(see, e.g., [8,9,10,11,12]), and weak nonlinear electromagnetic waves in ferromagnetic [13], antiferromagnetic[14], and dielectric[15] systems under external magnetic fields. Both of vanishing boundary conditions (VBC) and nonvanishing boundary conditions (NVBC) for the DNLS equation are physically significant. For problems of nonlinear Alfvén waves, weak nonlinear electromagnetic waves in magnetic and dielectric media, waves propagating strictly parallel to the ambient magnetic fields are modelled by VBC while those oblique waves are modelled by NVBC. In optical fibers, pulses under bright background waves are modelled by NVBC.

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It has been known that integrable systems admit soliton solutions which pass through each other in a completely elastic fashion. The only traces of their interactions are shifts of positions and phases. In some systems, besides “pure” solitons which keep their forms unchanged in transmissions, there exist bound states of these pure solitons called breathers which periodically oscillate (see, e.g., [16]).

Soliton solutions for the DNLS equation with VBC, including one-soliton solution[17] and multi-soliton formulas(e.g.,[18,19]), have been known. Researches on the DNLS equation with NVBC showed that its general one-soliton solution (corresponding to a complex discrete spectral parameter) is a breather which degenerates to a pure bright or dark soliton when the discrete spectral parameter becomes purely imaginary[20,21,4,5,22]. In known (1+1)-dimensional one-component integrable systems, the DNLS equation with NVBC is a rare instance which simultaneously supports pure bright solitons, pure dark solitons, as well as their bound states. Collisions between these solitons are thus important topics. However, like other NVBC problems, a double-valued function of the spectral parameter inevitably appears in the inverse scattering transform (IST) for the DNLS equation with NVBC, greatly complicating the IST. Early IST for the system performed on Riemann sheets only obtained modulus of a one-soliton solution[20] and asymptotic behaviors of the modulus of an implicit pure two-soliton solution[21]. Although the phase of the one-soliton solution was found later, yielding a very complicated solution[4,5], it is a too tedious task to get an explicit multi-soliton solution based on the IST performed on Riemann sheets [20,21]. A recent multi-soliton formula using Bäcklund transformation[23] was also unable to explicitly demonstrate collisions between solitons.

It has been suggested that constructing Riemann sheets for such NVBC problems can be avoided if one performs the IST on the plane of an appropriate affine parameter[24]. The technique was recently applied to the DNLS equation with NVBC, yielding not only a much simpler one-soliton solution than those in the literature but also a simple IST for further researches[22]. Immediately following Ref. [22], infinite number of conservation laws was derived by a simple standard procedure[25] and the evolution of a rectangular initial pulses in the system was considered, which was shown to be highly nontrivial and significantly different from all known results of other integrable systems[26].

It should be emphasized that these DNLS solitons might not exist in all of the physical systems mentioned above when the (1+1)-dimensional DNLS equation is not rigorously valid. For example, in space plasma, when the DNLS equation was generalized to a multi-dimensional one[27,28], it was shown that the dark solitons are not stable in presence of transverse perturbations[28].

In this letter, we only consider pure solitons of the DNLS equation with NVBC.

We modify the IST in Ref.[22] to the case when all discrete spectral parameters are purely imaginary and derive an explicit pure two-soliton solution. The solution consists of two bright solitons, two dark solitons, or one bright soliton and one dark soliton. Shifts of soliton positions due to collisions between them are obtained. We find that the collision between a bright soliton and a dark soliton is similar to usual collisions between two bright solitons or two dark solitons: the position of the faster one get a forward shift while the position of the slower one get a backward shift. We also show that the shifts of soliton positions are irrespective of the bright or dark characters of those participating solitons.

We write the DNLS equation as

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0, \quad (1)$$

where the subscript denotes partial derivative. Its first Lax equation[17] is

$$\partial_x F = LF, \quad (2)$$

with

$$L = -i\lambda^2 \sigma_3 + \lambda U, \quad (3)$$

$$U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}. \quad (4)$$

Here $\sigma_i (i = 1, 2, 3)$ are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5)$$

the bar stands for complex conjugate, and λ is the time-independent spectral parameter. As there is no phase shift across the DNLS solitons with NVBC[4,22], the NVBC can be simply written as

$$u \rightarrow \rho, \quad \text{as } x \rightarrow \pm\infty, \quad (6)$$

where ρ is real. Asymptotic solutions of Eq. (2) is

$$E^\pm(x, k) = (I - i\rho k^{-1} \sigma_1) e^{-i\lambda\zeta x \sigma_3}, \quad \text{as } x \rightarrow \pm\infty, \quad (7)$$

where we introduce an affine parameter k to make $\zeta = (\lambda^2 + \rho^2)^{\frac{1}{2}}$, a double-valued function of λ , become single-valued function of k , with

$$\lambda = \frac{1}{2}(k - \rho^2 k^{-1}), \quad \zeta = \frac{1}{2}(k + \rho^2 k^{-1}). \quad (8)$$

We define Jost solutions,

$$\Psi(x, k) \rightarrow E^+(x, k), \quad \text{as } x \rightarrow \infty, \quad (9)$$

$$\Phi(x, k) \rightarrow E^-(x, k), \quad \text{as } x \rightarrow -\infty, \quad (10)$$

where

$$\Psi(x, k) = (\tilde{\psi}(x, k), \psi(x, k)), \quad (11)$$

$$\Phi(x, k) = (\phi(x, k), \tilde{\phi}(x, k)), \quad (12)$$

and the scattering coefficients by

$$\phi(x, k) = a(k)\tilde{\psi}(x, k) + b(k)\psi(x, k), \quad (13)$$

$$\tilde{\phi}(x, k) = \tilde{a}(k)\psi(x, k) - \tilde{b}(k)\tilde{\psi}(x, k). \quad (14)$$

$\psi(x, k)$, $\phi(x, k)$ and $a(k)$ are analytic in the first and the third quadrants of the complex k plane, while $\tilde{\psi}(x, k)$, $\tilde{\phi}(x, k)$ and $\tilde{a}(k)$ are analytic in the second and the fourth quadrants.

As shown in Ref.[22], on the plane of the affine parameter k , if $k_{n1} = k_n$ is a simple zero of $a(k)$ in the first quadrant, then $k_{n2} = -k_n$, $k_{n3} = \rho^2 \bar{k}_n^{-1}$, and $k_{n4} = -\rho^2 \bar{k}_n^{-1}$ are also simple zeros. For the case when all discrete parameters $\lambda_n (n = 1, 2, \dots, N)$ are purely imaginary, all zeros of $a(k)$ locate on the circle of radius ρ centered at the origin(ρ -circle), that is

$$k_n = \rho \exp(i\beta_n), \quad 0 < \beta_n < \pi/2, \quad n = 1, 2, \dots, N. \quad (15)$$

Then, $k_{n3} = k_{n1}$, $k_{n4} = k_{n2}$, contributions of k_{n3} and k_{n4} must be dropped from relevant equations obtained in Ref.[22]. Therefore, for reflectionless potentials, we get

$$a(k) = e^{in} \prod_{n=1}^N \frac{k^2 - k_n^2}{k^2 - \bar{k}_n^2}, \quad (16)$$

where $\eta = -2 \sum_n \beta_n$, the inverse scattering equation,

$$\tilde{\psi}(x, k) e^{i\lambda\zeta x} = \begin{pmatrix} e^{-i\eta^+} \\ -i\rho k^{-1} e^{i\eta^+} \end{pmatrix} + 2 \sum_{n=1}^N \begin{pmatrix} k_n & 0 \\ 0 & k \end{pmatrix} \frac{c_n \psi(x, k_n)}{k^2 - k_n^2} e^{i\lambda_n \zeta_n x}, \quad (17)$$

and the expression for reflectionless potentials (solitons),

$$u(x) = \rho e^{-i2\eta^+} - 2\rho e^{-i\eta^+} \sum_{n=1}^N \frac{c_n}{k_n} \psi_1(x, k_n) e^{i\lambda_n \zeta_n x}. \quad (18)$$

Here

$$\begin{aligned} \lambda_n &= i\rho \sin \beta_n, & \zeta_n &= \rho \cos \beta_n, \\ \nu_n &= \rho^2 \sin(2\beta_n), & v_n &= \rho^2(1 + 2 \sin^2 \beta_n), \end{aligned}$$

$$c_n(t) = c_n(0) e^{i2\lambda_n \zeta_n (2\lambda_n^2 - \rho^2)t} = c_n(0) e^{\nu_n v_n t}, \quad (19)$$

and

$$\eta^+(x) = \frac{1}{2} \int_x^\infty (\rho^2 - |u|^2) dx. \quad (20)$$

Symmetric relations such as

$$\tilde{\psi}(x, \rho^2 k^{-1}) = i\rho^{-1} k \sigma_3 \psi(x, k), \quad (21)$$

found in Ref.[22] are still valid. At $k = k_n$, we have

$$\tilde{\psi}(x, \bar{k}_n) = i\rho^{-1} k_n \sigma_3 \psi(x, k_n), \quad (22)$$

then, at $k = \bar{k}_m$, the first component of Eq.(17) is

$$ie^{i\beta_m} \psi_1(x, k_m) = e^{-i\eta^+} e^{i\lambda_m \zeta_m x} + 2 \sum_{n=1}^N \frac{k_n c_n \psi_1(x, k_n)}{\bar{k}_m^2 - k_n^2} e^{i(\lambda_m \zeta_m + \lambda_n \zeta_n)x}. \quad (23)$$

It can be shown that for k_n located on the ρ -circle,

$$c_n = ie^{i\beta_n} \times (\text{a real number}). \quad (24)$$

We can set

$$c_n(0) = i\chi_n \rho \sin(2\beta_n) e^{i\beta_n} e^{\nu_n x_n}, \quad \chi_n = \pm 1. \quad (25)$$

We also define

$$\theta_n = \nu_n(x - x_n - v_n t). \quad (26)$$

In principle, one can find $\psi_1(x, k_m)$ by solving linear equations Eq.(23) and then get a multi-soliton solution with Eq.(20) and Eq.(18). We only consider one-soliton and two-soliton solutions in this letter.

For the case of $N = 1$, we get

$$u_1 = \rho e^{-i2\eta_1^+} \frac{A_1}{D_1}, \quad (27)$$

where

$$D_1 = 1 - i\chi_1 e^{i\beta_1} e^{-\theta_1}, \quad (28)$$

$$A_1 = 1 - i\chi_1 e^{-i3\beta_1} e^{-\theta_1}, \quad (29)$$

$$\eta_1^+ = \frac{1}{2} \int_x^\infty (\rho^2 - |u_1|^2) dx = i \ln \frac{D_1}{\bar{D}_1}, \quad (30)$$

and thus the one-soliton solution,

$$u_1 = \rho \frac{A_1 D_1}{\bar{D}_1^2} = u_1(\theta_1), \quad (31)$$

which is identical to that obtained in the literature. It is a bright soliton for $\chi_1 = -1$ or a dark soliton for $\chi_1 = 1$. There is only one parameter, β_1 , characterizing the soliton which is usually called one-parameter soliton[5].

For the case of $N = 2$, we get

$$u_2 = \rho e^{-i2\eta_2^+} \frac{A_2}{D_2}, \quad (32)$$

where

$$\begin{aligned} D_2 &= 1 - i\chi_1 e^{i\beta_1} e^{-\theta_1} - i\chi_2 e^{i\beta_2} e^{-\theta_2} \\ &\quad - \chi_1 \chi_2 \frac{\sin^2(\beta_1 - \beta_2)}{\sin^2(\beta_1 + \beta_2)} e^{i(\beta_1 + \beta_2)} e^{-\theta_1 - \theta_2}, \end{aligned} \quad (33)$$

$$A_2 = 1 - i\chi_1 e^{-i3\beta_1} e^{-\theta_1} - i\chi_2 e^{-i3\beta_2} e^{-\theta_2} - \chi_1 \chi_2 \frac{\sin^2(\beta_1 - \beta_2)}{\sin^2(\beta_1 + \beta_2)} e^{-i3(\beta_1 + \beta_2)} e^{-\theta_1 - \theta_2}. \quad (34)$$

In order to find η_2^+ , we find

$$\text{Re}(D_2) \frac{d[\text{Im}(D_2)]}{dx} - \text{Im}(D_2) \frac{d[\text{Re}(D_2)]}{dx} = \frac{\rho^2}{4} (|D_2|^2 - |A_2|^2). \quad (35)$$

With this relation, we get

$$\eta_2^+ = \frac{1}{2} \int_x^\infty (\rho^2 - |u_2|^2) dx = i \ln \frac{D_2}{\bar{D}_2}, \quad (36)$$

and the two-soliton solution,

$$u_2 = \rho \frac{A_2 D_2}{\bar{D}_2^2}. \quad (37)$$

Assume $\beta_2 > \beta_1$. At times long before collision ($t \rightarrow -\infty$), in the vicinity of $\theta_1 \approx 0$, $\theta_2 \rightarrow \infty$, $u_2 \approx u_1(\theta_1)$, while in the vicinity of $\theta_2 \approx 0$, $\theta_1 \rightarrow -\infty$, $u_2 \approx u_1(\theta_2 + \Delta)$, that is,

$$u_2 \approx u_1(\theta_1) + u_1(\theta_2 + \Delta), \quad (38)$$

where

$$\Delta = 2 \ln \left| \frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)} \right| > 0. \quad (39)$$

u_2 consists of two well separated solitons, moving to the positive direction of the x -axis, with the slower soliton of β_1 moving on the front.

At times long after collision ($t \rightarrow \infty$), in the vicinity of $\theta_1 \approx 0$, $\theta_2 \rightarrow -\infty$, $u_2 \approx u_1(\theta_1 + \Delta)$, while in the vicinity of $\theta_2 \approx 0$, $\theta_1 \rightarrow \infty$, $u_2 \approx u_1(\theta_2)$, that is,

$$u_2 \approx u_1(\theta_1 + \Delta) + u_1(\theta_2). \quad (40)$$

Now u_2 consists of two well separated solitons, with the faster soliton of β_2 moving ahead. These asymptotic solutions show that the faster soliton of β_2 catches up the soliton of β_1 , collides with the latter and leaves it behind, gets a forward shift $\Delta x_2 = \Delta/\nu_2$ in position while the soliton of β_1 gets a backward shift $\Delta x_1 = -\Delta/\nu_1$ in position. These shifts in position are independent of

χ_1 and χ_2 , i.e., the bright or dark characters of the two participating solitons. They are in agreement with those in Ref.[21] obtained by discussing the asymptotic behavior of the modulus of an implicit two-soliton solution.

Collisions between two bright solitons(Fig.1), two dark solitons (Fig.2), and one bright soliton and one dark soliton(Fig.3 and Fig.4) are graphically shown, where shifts of soliton positions due to collisions are vividly seen. We choose $x_1 = \Delta/(2\nu_1)$ and $x_2 = \Delta/(2\nu_2)$ in these figures so that the two solitons completely overlap at $(x, t) = (0, 0)$.

In summary, we find an explicit two-soliton solution for the DNLS equation with NVBC. Shifts of soliton positions due to collisions between solitons are analytically obtained. Details of collisions between two bright solitons, two dark solitons and one bright soliton and one dark soliton are graphically shown. It is interesting to note that these shifts only depend on parameters of the participating solitons, irrespective of their bright or dark characters. Observing Eq.(30) and Eq.(36), one can find that the relations between η and D for one-soliton and two-soliton are the same. The relation can possibly be extended to multi-soliton case and be helpful to find an explicit multi-soliton solution.

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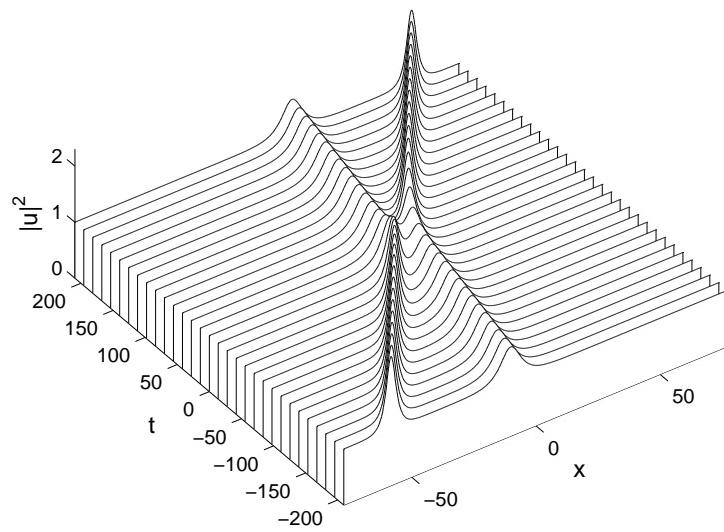


Fig. 1. Collision between two bright solitons, $\rho = 1$, $\beta_1 = \pi/12$, $\beta_2 = \pi/24$. Variables in the figure are dimensionless.

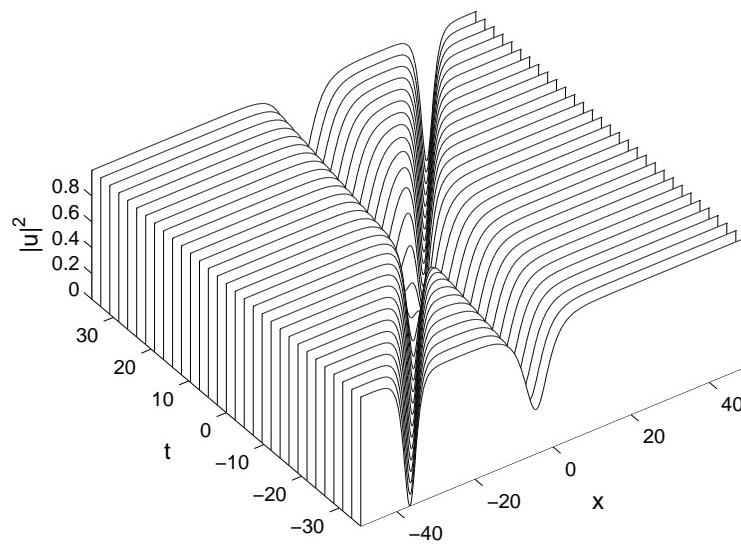


Fig. 2. Collision between two dark solitons, $\rho = 1$, $\beta_1 = \pi/15$, $\beta_2 = \pi/6$. Variables in the figure are dimensionless.

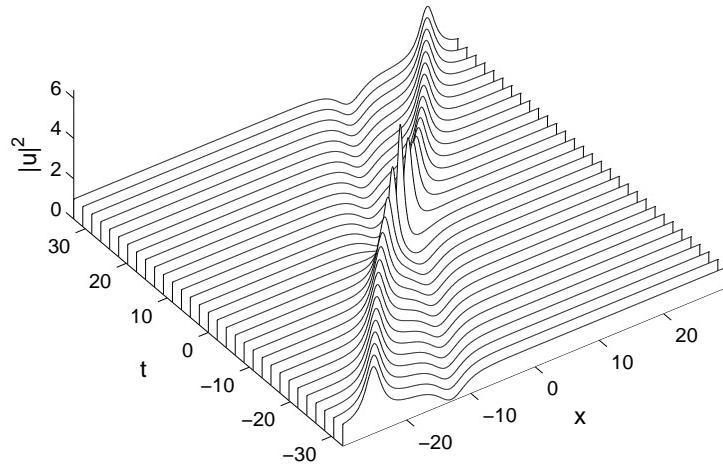


Fig. 3. A bright soliton of $\beta_2 = 2\pi/15$ chases a dark soliton of $\beta_1 = 7\pi/60$, $\rho = 1$. Variables in the figure are dimensionless.

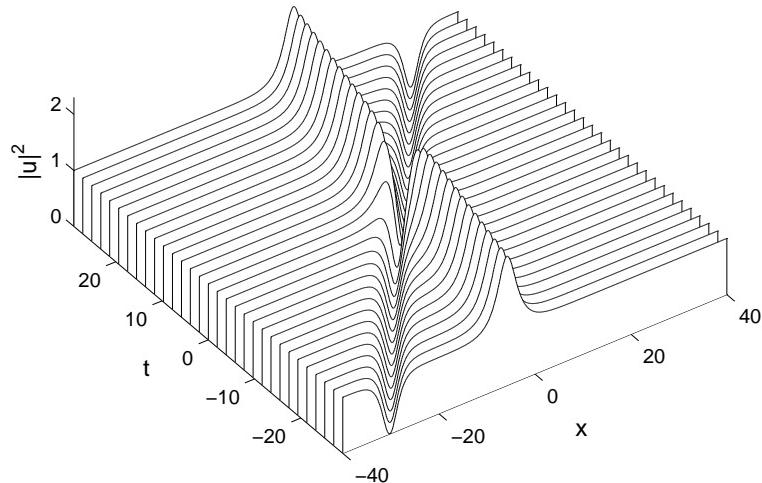


Fig. 4. A dark soliton of $\beta_2 = \pi/6$ chases a bright soliton of $\beta_1 = \pi/12$, $\rho = 1$. Variables in the figure are dimensionless.